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The goals of this research project were the development of control and estimation algorithms for nonlinear systems which are computationally feasible with robust performance despite numerical and modeling errors. The approach was based on the recent generalization of linear worst case (H-infinity) controllers to nonlinear systems. The construction of nonlinear H-infinity controllers depends on the solution of two PDE's of Hamilton-Jacobi type. The first is the one associated with the problem of H-infinity suboptimal control by state feedback that has appeared previously in the work of several authors. Numerical methods to compute a Taylor series solution term by term have been developed. The second PDE is a new Hamilton-Jacobi equation associated with H-infinity suboptimal estimation. A hybrid computational method to solve such problems has been developed.

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Final Technical Report
Computational Nonlinear Control
AFOSR-95-1-0169

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Abstract

The goals of this research project were the development of control and estimation algorithms for nonlinear systems which are computationally feasible with robust performance despite numerical and modeling errors. The approach was based on the recent generalization of linear worst case (H_∞) controllers to nonlinear systems. The construction of nonlinear H_∞ controllers depends on the solution of two PDE's of Hamilton-Jacobi type. The first is the one associated with the problem of H_∞ suboptimal control by state feedback that has appeared previously in the work of several authors. Numerical methods to compute a Taylor series solution term by term have been developed. The second PDE is a new Hamilton-Jacobi equation associated with H_∞ suboptimal estimation. A hybrid computational method to solve such problems has been developed .

1 Nonlinear H_∞ Control by State Feedback

In this section an algorithm for computing term by term a state feedback H_∞ control law for a nonlinear system is described. The MATLAB code to impliment this algorithm is available from http://scad.utdallas.edu/scad/ in the Nonlinear Systems Toolbox. The function "hji.m" impliments the following algorithm.

Consider a nonlinear system of the form

$$\dot{x} = f(x, u). \tag{1}$$

where the input u consists of two subvectors $u = [u_c; u_d]$, a control u_c and a disturbance u_d. The goal is to compute the feedback law for u_c that minimizes the maximum over all u_d the cost

$$\int_0^\infty l(x, u)\ dt. \tag{2}$$

See [K1] for further details. It is assumed that for each x, l is strictly convex in u_c, strictly concave in u_d.

When it exists and is smooth the optimal cost $\pi(x)$ starting at x will satisfy the Hamilton-Jacobi-Isaacs PDE

$$\begin{aligned} \Pi_x\, f + l &= 0 \qquad (3) \\ \Pi_x\, f_u + l_u &= 0 \qquad (4) \end{aligned}$$

where for each x the evaluation is at the minmax u satisfying (4).

It is further assumed that f, l have series expansions

$$\begin{aligned} f(x,u) &= f^{[1]}(x,u) + f^{[2]}(x,u) + f^{[3]}(x,u) + \ldots + f^{[d]}(x,u) \qquad (5) \\ l(x,u) &= l^{[2]}(x,u) + l^{[3]}(x,u) + l^{[4]}(x,u) + \ldots + l^{[d+1]}(x,u) \qquad (6) \end{aligned}$$

where $f^{[j]}(x,u)$, $l^{[j]}(x,u)$ denote homogeneous polynomials of degree j. The leading terms are of the familiar linear-quadratic form

$$\begin{aligned} f^{[1]}(x,u) &= Ax + Bu \qquad (7) \\ l^{[2]}(x,u) &= \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right). \qquad (8) \end{aligned}$$

The matrix R is symmetric and invertible but not positive definite because of the assumption that l is strictly convex in u_c, strictly concave in u_d.

Following Al'brekht [A], it is assumed that the optimal cost $\pi(x)$ and feedback $\kappa(x)$ have similar power series expansions

$$\begin{aligned} \pi(x) &= \pi^{[2]}(x) + \pi^{[3]}(x) + \pi^{[4]}(x) + \ldots + \pi^{[d+1]}(x) \qquad (9) \\ u &= \begin{bmatrix} u_c \\ u_d \end{bmatrix} = \kappa(x) = \kappa^{[1]}(x) + \kappa^{[2]}(x) + \kappa^{[3]}(x) + \ldots + \kappa^{[d]}(x) \qquad (10) \end{aligned}$$

where the leading terms are

$$\begin{aligned} \pi^{[2]}(x) &= \frac{1}{2}x'Px \qquad (11) \\ \kappa^{[1]}(x) &= Kx \qquad (12) \end{aligned}$$

Note that the feedback is the optimal one for both the control u_c and the disturbance u_d.

These expansions are plugged into the Hamilton-Jacobi-Isaacs equations and terms of the same degree are collected. The leading terms in (3) are degree 2 and those in (4) are of degree 1, and yield the familiar Riccati equation and state feedback,

$$\begin{aligned} 0 &= PA + A'P + Q - (PB + S)R^{-1}(B'P + S') \qquad (13) \\ K &= -R^{-1}(B'P + S') \qquad (14) \end{aligned}$$

The next step is to collect terms in (3) of degree 3 and those in (4) of degree 2, this yields a system of linear equations for $\pi^{[3]}$, $\kappa^{[2]}$ involving P, K which are then solved. The process is repeated through terms in (3) of degree $d+1$ and those in (4) of degree d yielding linear equations for $\pi^{[d+1]}$, $\kappa^{[d]}$ that depend on the lower degree solutions.

The call of "hji.m" is of the form

$$[ka, fk, py, lk] = hji(f, l, n, m, d);$$

The input parameters f, l are the dynamics and lagrangian as above. More precisely, they are matrices of coefficients of the various monomials in f, l in block lexographic order as described in the Read Me First file in the Nonlinear Systems Toolbox. The parameters n, m are the dimensions of x, u respectively and d is the degree of the series expansion of f.

The first output parameter is $ka = \kappa$ and the third output parameter is $py = \pi$ as described above. More precisely they are matrices of their coefficients in block lexographic order. The other output parameters are the matrices of the coefficients of the closed loop dynamics $fk(x) = f(x, \kappa(x))$ and the closed loop lagrangian $lk(x) = l(x, \kappa(x))$

The routine "hji.m" differs from "hjb.m" by calling "care.m" instead of "lqr2.m" to solve the Riccati equation. The routine "care.m" can handle indefinte Q and R while "lqr2.m" requires that Q be nonnegative definite and R be positive definite. The current release of the MATLAB Control Toolbox contains "lqr2.m" and the next release will contain "care.m".

Both "hji.m" and "hjb.m" can solve the Hamilton-Jacobi PDE for dynamics and lagrangians that depend on static or dynamic parameters. Such problems arise in designing servos [K2] and model matching controllers [K3]. Certain restrictions apply. See the comments in the m-files.

2 Hybrid Nonlinear Estimation

A hybrid algorithm for a nonlinear state observer that utilizes two levels of computation has been developed [KrD]. On the higher level one approximately computes a negative log liklihood function $Q(x, t)$ for the currnt state given the past obseravtions and initial state liklihood. The most likely estimate of $x(t)$ is $\hat{x}(t) = \text{argmin} Q(x, t)$ At the lower level, we initiate local observers that resemble extended Kalman filters at the local minima of $Q(x, t)$. These are computed on a much faster time scale. One also computes how well they explain the observations. and takes as the estimate, the one that best explains the observations to date. This algorithm can be suitably modified to calculate nonlinear H_∞ estimators [K1].

Since the computation of the Q function is expensive, it is done on a relatively coarse spatial and temporal grid. Hence the minimum of Q converges slowly to the true state and is never very accurate due to the coarseness of the grid. The local observers are computationally inexpensive especially since the filter gains are derived from Q rather than solutions of Riccati equations. Moreover when initialized close to the true value, they converge quickly and accurately. However if they are initialized far from the true value, they don't always converge to it. The coarse information in Q allows one to initialize the local observers properly.

Mortensen [M] and Hijab [H] introduced the concept of minimum energy estimation. Given an initial state estimate $\hat{x}^0$, an observation history $\{y(s) : 0 \leq s \leq t\}$ and an endpoint x one seeks the minimum "energy" triple x^0, $w(s)$, $v(s)$ satisfying

$$\dot{x}(s) = f(x(s)) + w(s) \tag{15}$$

$$y(s) = h(x(s)) + v(s) \tag{16}$$

$$x(0) = x^0 \tag{17}$$
$$x(t) = x. \tag{18}$$

The "energy" of the triple x^0, $w(s)$, $v(s)$ is defined as

$$\frac{1}{2}\int_0^t e^{-\alpha(t-s)} \left| \begin{matrix} w(s) \\ v(s) \end{matrix} \right|^2 ds + \frac{e^{-\alpha t}}{2} \left| \hat{x}^0 - x^0 \right|^2. \tag{19}$$

Increasing the forgetting factor α decreases the importance of the initial state estimate and earlier observations and increases the importance of the later observations. The value $Q(x,t)$ is a measure of the likelihood that $x(t) = x$ given the initial state estimate and the observations to date. The smaller $Q(x,t)$ is, the more likely $x(t) = x$. Let $Q(x,t)$ denote the infimum of (19) over all triples satisfying (15-18), then the minimum energy estimate is

$$\hat{x}(t) = \text{argmin} Q(x,t). \tag{20}$$

It is not hard to see that Q is a solution in the viscosity sense of the Hamilton Jacobi Bellman (HJB) PDE

$$\alpha Q + Q_t + Q_x f + \frac{1}{2} Q_x Q_x' - \frac{1}{2}|y-h|^2 = 0. \tag{21}$$

For H_∞ estimation the PDE is slightly different, [K1],

$$Q_t + Q_x\, f + \frac{1}{2\gamma^2} Q_x Q_x' - \frac{\gamma^2}{2}\,|y-c|^2 + \frac{1}{2}\,|k-\hat{u}|^2 = 0 \tag{22}$$

Following Kushner and Dupuis [KD], we compute Q not by approximately solving the HJB PDE but rather by solving an approximating nonlinear program. Let r, k be relatively coarse spatial and temporal steps. Choose a subdomain of $\mathbf{R^n}$ where the state is known to be and consider the rectangular lattice of points in the subdomain with spacing r. Following the dynamic program approach (in forward time), we define the approximate solution $Q(x,t)$ of (21) at lattice points x and time steps t by

$$\begin{aligned} Q(x,t+k) &= \inf \{(1-\alpha k) Q(z,t) \\ &\quad + \left| \frac{x-z}{k} - \frac{f(x,t+k)+f(z,t)}{2} \right|^2 \frac{k}{2} \\ &\quad + \left| y(t) - \frac{h(x,t)+h(z,t)}{2} \right|^2 \frac{k}{2} \Bigg\} \end{aligned} \tag{23}$$

$$Q(x,0) = \frac{1}{2}\left| x - \hat{x}^0 \right|^2 \tag{24}$$

where the infimum is over z in the whole lattice in the subdomain or some subset such as the $2n$ nearest neighbors of x.

Notice that the computation of Q must be done in real time because of the presence of $y(t)$. The complexity of the computation is inversly proportional to the spatial step r to the power of the state dimension n. Hence there is a tradeoff between accuracy (small r) and computational ease (large r). Of course similar difficulties arise in all nonlinear estimation

algorithms, for example, nonlinear filtering requires solving the Zakai stochastic PDE in real time.

The extended Kalman Filtering is an alternative approach which can be very accurate when it converges. However it may fail to converge if the problem is highly nonlinear. If we assume that the w, v in (15,16) are independent standard white Gaussian noises and the initial state estimate is an independent Gaussian random vector with mean $\hat{x}^0$ and covariance P^0 then the extended Kalman Filter (EKF) takes the form

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x},t) + Ph_x'(\hat{x},t)(y - h(\hat{x},t)) && (25)\\
\dot{P} &= f_x(\hat{x},t)P + Pf_x(\hat{x},t)' + I \\
&\quad -Ph_x'(\hat{x},t)h_x(\hat{x},t)P && (26)\\
\hat{x}(0) &= \hat{x}^0 && (27)\\
P(0) &= P^0 && (28)
\end{aligned}$$

An example of a highly nonlinear problem where an EKF may fail to converge is

$$\begin{aligned}
\dot{x} &= x(1 - x^2) && (29)\\
y &= x^2 + \epsilon x. && (30)
\end{aligned}$$

If $\epsilon = 0$ the states x, $-x$ are indistinguishable but for nonzero ϵ the system is observable. The dynamics has stable equilibria at $x = \pm 1$ and an unstable equilibrium at $x = 0$. If $\epsilon > 0$, the system is initialized near -1 and the EKF is initialized near 1, the EKF will fail to converge to the true value [KrD].

Suppose $Q(x,t)$ is a smooth solution to HJB PDE (21), $\hat{x}(t)$ is a relative minimum of $Q(x,t)$ and $q(t) = Q(\hat{x}(t),t)$ then

$$\begin{aligned}
0 &= \frac{\partial Q}{\partial x}(\hat{x}(t),t) && (31)\\
0 &= Q_{xx}(\hat{x}(t),t)\dot{\hat{x}}(t) + Q_{xt}(\hat{x}(t),t). && (32)
\end{aligned}$$

If one partially differentiates (21) with respect to x and evaluates at $\hat{x}(t)$, t one obtains

$$\begin{aligned}
0 &= Q_{tx}(\hat{x}(t),t) + Q_{xx}(\hat{x}(t),t)f \\
&\quad +h_x'(\hat{x}(t),t)(y - h(\hat{x}(t),t). && (33)
\end{aligned}$$

From the last two equations one obtains

$$\dot{\hat{x}} = f(\hat{x},t) + Q_{xx}^{-1}(\hat{x},t)h_x'(\hat{x},t)(y - h(\hat{x},t)) \quad (34)$$

and evaluating (21) at $\hat{x}(t)$, t yields

$$\dot{q} = -\alpha\, q + \frac{1}{2}|y - h(\hat{x},t)|. \quad (35)$$

These are the equations of a local observer based on Q. Notice the similarity of (34) to (25) of an EKF.

The hybid approach is as follows.
1) Compute $Q(x,t)$ by a nonlinear programming approximation (23) on a coarse spatial and temporal grid,
2) At each relative minimum of Q(x,t), initialize a local observer $\hat{x}(t), q(t)$
3) Let the various local observers $\hat{x}(t), q(t)$ evolve according to (34, 35) on a fast time scale,
4) Eliminate redundant local observers when they come close together,
5) Choose as the current estimate, the $\hat{x}(t)$ of the local observer with smallest $q(t)$.

While this algorithm can result in a large number of local observers, the computational burden associated with each one is quite small, less than an EKF. Each local observer invloves integrating $n+1$ diffferential equations instead of $(n^2+3n)/2$ for an EKF. The total computational burden associated with computing $Q(x,t)$ on a coarse spatial and temporal grid and computing many local observers on a fine temporal scale is considerably lighter than computing $Q(x,t)$ on a fine spatial and temporal grid. Moreover the accuracy of the solution of the HJB PDE (21) is limited by the fineness of the spatial grid while machine precision is the limit on the spatial accuracy of a local observer. In [KrD] this observer is applied to a pair of examples.

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2. A. J. Krener and A. Duarte, *A Hybrid computational approach to nonlinear estimation*, Proc. of 1996 Conference on Decision and Control

3. A. J. Krener, *Nonlinear Systems Toolbox V. 1.0*, (1996) A MATLAB based toolbox available from http://scad.utdallas.edu/scad/

REPORT OF INVENTIONS AND SUBCONTRACTS

(Pursuant to "Patent Rights" Contract Clause) (See Instructions on Reverse Side.)

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a. NAME(S) OF INVENTOR(S) (Last, First, MI)	b. TITLE OF INVENTION(S)	c. DISCLOSURE NO., PATENT APPLICATION SERIAL NO. OR PATENT NO.	d. ELECTION TO FILE PATENT APPLICATIONS				e. CONFIRMATORY INSTRUMENT OR ASSIGNMENT FORWARDED TO CONTRACTING OFFICER	
			(1) United States		(2) Foreign			
			(a) Yes	(b) No	(a) Yes	(b) No	(1) Yes	(
KRENER, Arthur J.	none	--	--	--	--	--	--	--

f. EMPLOYER OF INVENTOR(S) NOT EMPLOYED BY CONTRACTOR/SUBCONTRACTOR		g. ELECTED FOREIGN COUNTRIES IN WHICH A PATENT APPLICATION WILL BE FILED	
(1) (a) Name of Inventor (Last, First, MI)	(2) (a) Name of Inventor (Last, First, MI)	(1) Title of Invention	(2) Foreign Countries of Patent Application
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(b) Name of Employer	(b) Name of Employer		
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SECTION II – SUBCONTRACTS (Containing a "Patent Rights" clause)

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SECTION III – CERTIFICATION

7. CERTIFICATION OF REPORT BY CONTRACTOR/SUBCONTRACTOR (Not required if ☐ Small Business or ☐ Non-Profit organization.) (X appropriate box)

a. NAME OF AUTHORIZED CONTRACTOR/SUBCONTRACTOR OFFICIAL (Last, First, MI)	c. I certify that the reporting party has procedures for prompt identification and timely disclosure of "Subject Inventions," that such procedures have been followed and that all "Subject Inventions" have been reported.	
Arthur J. Krener		
b. TITLE	d. SIGNATURE	e. DATE SIGNED
Professor		01/03/97

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